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CANADIAN THESES ON MICROFICHE | THÈSES CANADIENNES SUR MICROFICHE

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TITLE OF THESIS/*TITRE DE LA THÈSE* The Classification of Monotone Analytic Functions on Surfaces

UNIVERSITY/*UNIVERSITÉ* Univ. of Alberta

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* M.Sc.

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE GRADE* 1978

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THE UNIVERSITY OF ALBERTA

THE CLASSIFICATION OF MONOTONE ANALYTIC FUNCTIONS ON SURFACES

by

CHRISTOPHER SMALL

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

THE DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

SPRING, 1978

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled THE CLASSIFICATION OF MONOTONE ANALYTIC FUNCTIONS ON SURFACES submitted by CHRISTOPHER SMALL in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

The work of analytic topologists, such as Whyburn, has shown that under certain conditions a mapping may be factored into the composition of a monotone mapping followed by a light mapping. A monotone mapping is one for which the point inverses are continua. A light mapping is a mapping for which the point inverses are totally disconnected sets. This factorization can also be accomplished under more restrictive conditions in the categories of differential and analytic manifolds with differential and analytic maps respectively.

In this thesis, monotone real valued analytic mappings on surfaces are studied, and classification theorems are developed for such mappings on the sphere, torus and projective plane. It is shown that there exist exactly two topologically distinct monotone real valued analytic mappings on the sphere. On the torus, there are shown to be at most nine such mappings, and exactly four on the projective plane.

ACKNOWLEDGEMENTS

I would like to thank my first supervisor, Dr. J.G. Timourian for his help, and for his patience in dealing with an easily discouraged student.

I would also like to thank my second supervisor, Dr. G.J. Butler, who helped change an abstract possibility into a concrete reality.

Finally, I am grateful to the National Research Council of Canada, which gave me financial assistance while I was working on this thesis.

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CHAPTER I

INTRODUCTION

The notion of monotonicity of maps between topological spaces is a generalization of the concept of monotonicity for functions from the real numbers to themselves. Let X and Y be spaces. A mapping, f, from X to Y, such that for all $x \in X$, $f^{-1}f(x)$ is compact and connected is said to be monotone. The concept in this form is due to Morrey [13].

It has been shown possible to classify the images of certain classes of spaces under monotone mappings. An example of this is the classification of such images for surfaces without boundary. Such classification theorems for monotone mappings are relevant to the study of mappings of a more general nature. As will be shown in the next chapter, a mapping, under certain conditions, can be factored as $\ell \circ m$, where m is a monotone mapping and ℓ is a light mapping (i.e., the inverse image of every value of ℓ is totally disconnected).

We will limit our study of monotone mappings to those which are real valued analytic mappings on analytic surfaces. In particular, the sphere S^2, the torus T^2, and the projective plane P^2, defined as

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

$$T^2 = \{(x,y,z,w) \in \mathbb{R}^4 : x^2 + y^2 = 1,\ z^2 + w^2 = 1\}$$

$$P^2 = S^2/(x,y,z) \sim (-x, -y, -z)$$

will be considered. The space P^2 is homeomorphic to a disc wrapped twice along its boundary around a circle. Let D^2 be a closed disc and ∂D^2 its boundary. Let S^1 be a circle disjoint from D^2, and let $g: \partial D^2 \to S^1$ be a two fold wrapping of ∂D^2 around S^1. Then $D^2 \cup_g S^1 = D^2 \cup S^1/x \sim g(x)$ is homeomorphic to P^2.

Examples of Monotone Mappings on S^2, P^2, T^2

1. Let X be a compact connected space, and c a real number. Let $f(X) = c$.

2. Define $f: S^2 \to \mathbb{R}$ by $f(x,y,z) = x$.

3. Let $f: P^2 \to \mathbb{R}$ be the function induced by the quotient map $q: S^2 \to P^2$ and the function $g: S^2 \to \mathbb{R}$ defined by $g(x,y,z) = z^2$.

4. Let $f: P^2 \to \mathbb{R}$ be induced as above by $g(x,y,z) = xz$.

5. Parameterize T^2 as (θ,ϕ), where (θ,ϕ) corresponds to $(\cos\theta, \sin\theta, \cos\phi, \sin\phi) \in \mathbb{R}^4$. Let $f: T^2 \to \mathbb{R}$ be defined by $f(\theta,\phi) = (1 + \sin\theta)(1 + \sin\phi)$.

6. Define $f: T^2 \to \mathbb{R}$ by $f(\theta,\phi) = \cos\theta(1 + \sin\phi)$.

Def. 1.1 Let X, X', Y and Y' be topological spaces. Let f be a mapping from X to X', and g a mapping from Y to Y'. The mapping f is said to be topologically equivalent to g if there exist two homeomorphisms $\alpha: X \to Y$, and $\beta: X' \to Y'$ such that $\beta \circ f = g \circ h$.

The homeomorphisms, α and β, "carry" this topological equivalence across from one mapping to the other.

It will be shown in Chapter 6 that there are exactly two topologically distinct real valued analytic monotone functions on S^2. On P^2, it will be shown that there are exactly four, and on T^2, at most nine.

Remark. It should be noted that analyticity cannot be replaced by differentiability in the classification stated above. The analytic functions cannot be constant on an open set without being constant everywhere. However, if $f: S^2 \to \mathbb{R}$ is defined piecewise by $f(x,y,z) = \exp\left(-\frac{1}{x^2}\right)$ for $x > 0$ and $f(x,y,z) = 0$ for $x \leq 0$, then f is indefinitely differentiable at all points in S^2.

CHAPTER II

MONOTONE FUNCTIONS AND FACTORIZATION

Note. In this chapter, all spaces introduced will be metric spaces.

Prop. 2.1. Let f be a mapping of X onto Y, where X is compact. Then f is monotone if and only if $f^{-1}(A)$ is connected for every connected set $A \subset Y$.

Pf. It is immediate that if $f^{-1}(A)$ is connected for every connected set A, then f is monotone.

Suppose f is monotone, and A is connected in Y. Suppose, also, that $f^{-1}(A) = U \cup V$ is a disconnection of $f^{-1}(A)$. Then for each $p \in A$, $f^{-1}(p)$ is contained in U or in V because it is connected. It follows that $f(U)$ and $f(V)$ are disjoint. Because $f(U) \cup f(V)$ cannot be a separation of A, there must exist a point y in either $f(U)$ or $f(V)$, say $f(U)$, which is the limit of a sequence of points (y_i) in $f(V)$. Choose (x_i), a sequence in V such that $f(x_i) = y_i$. Because X is compact, (x_i) must have an accumulation point, x. Then, $f(x) = y$ which implies that $x \in U$, which cannot be. Therefore $f^{-1}(A)$ must be connected. □

Lemma 2.2 Let X be separable, and let D be an upper semicontinuous decomposition of X into compact sets. If X/D is the decomposition space of X, then X/D is a separable metric space.

Pf. Whyburn ([23], pp. 123-125). □

Def. 2.3 Let (A_i) be a sequence of sets in a space X. Let $\limsup (A_i)$ be the set of all $x \in X$ for which every neighborhood of x meets infinitely many sets A_i. Let $\liminf (A_i)$ be the set of all $x \in X$ for which every neighborhood of x meets all but a finite number of the sets A_i. If $\liminf (A_i) = \limsup (A_i)$, then the limit of the sequence, $\lim (A_i) = \limsup (A_i) = \liminf (A_i)$, is said to exist.

Lemma 2.4 Let (A_i) be a sequence of connected sets in a compact space. Suppose $\lim (A_i)$ exists. Then $\lim (A_i)$ is connected.

Pf. Whyburn ([23], pp. 14-15). □

Lemma 2.5 Let (A_i) be a sequence of sets in a space. Then there exists a subsequence of (A_i) which has a limit.

Pf. Whyburn ([23], pp. 11-12). □

Lemma 2.6 Let X and Y be compact spaces, and let f be a mapping of X onto Y. For each $y \in Y$, let D_y be the decomposition of $f^{-1}(y)$ into its components. Then $D = \bigcup_{y \in Y} D_y$ is an upper semi-continuous decomposition of X.

Pf. Suppose D is not upper semicontinuous. Then there exists an $A \in D$, and a set U, open in X such that $A \subset U$, and for every open set V, where $A \subset V \subset U$, there exists some set $B \in D$ such that $V \cap B \neq \phi$ and $B \cap (X-U) \neq \phi$.

Let V_i be the set of all $x \in X$ closer than $\frac{1}{i}$ to A, where i is a positive integer. Because A is compact, for sufficiently large i, $V_i \subset U$. Assume without loss of generality that this is true

for all i. Then for each i, there exists a set $B_i \in D$ such that $V_i \cap B_i \neq \phi$ and $B_i \cap (X-U) \neq \phi$. By Lemma 2.5, it can be assumed without loss of generality that $\lim (B_i) = L$ exists. Let $f(A) = y$, and $f(B_i) = y_i$. Because $A \cap \lim\inf (B_i) \neq \phi$, it follows that (y_i) converges to y in Y. This implies that $L \subset f^{-1}(y)$. However, by Lemma 2.4, L must be connected, and therefore $L \subset A$. However, this contradicts the fact that $B_i \cap (X-U) \neq \phi$ for all i, because $\bigcup_{i=1}^{\infty} B_i \cap (X-U)$ must have an accumulation point, since X is compact.

Therefore D is upper semicontinuous. □

Prop. 2.7 Let X and Y be compact spaces, and suppose f is a mapping of X onto Y. Then there exist a metric space, Z, a monotone mapping g of X onto Z, a light mapping, h, of Z onto Y such that $f = h \circ g$. This factorization of f is unique.

Pf. Let D be the decomposition of X into the components of sets of the form $f^{-1}(y)$, $y \in Y$. It follows from Lemma 2.6 that D is upper semicontinous. Then Lemma 2.2 implies that the decomposition space of X, X/D is a metric space. Let Z be X/D, and let g be the quotient map from X to Z. It is immediate that g is monotone.

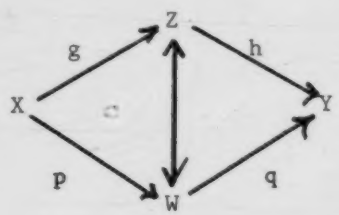
Define $h: Z \to Y$ by $h = f \circ g^{-1}$. To see that h is continuous, consider a sequence of points (z_i) converging to z in Z. Then $\lim\sup g^{-1}(z_i) \subset g^{-1}(z)$. Therefore, all except a finite number of the sets $g^{-1}(z_i)$ are in an arbitrary neighborhood of $g^{-1}(z)$. Thus $f \circ g^{-1}(z_i) = h(z_i)$ converges to $f \circ g^{-1}(z) = h(z)$. This implies that h is continuous.

The mapping, h, is also light. Let K be any component in Z of $h^{-1}(y)$, $y \in Y$. Because g is monotone, it follows that $g^{-1}(K)$ is connected, and therefore K must be a single component of $f^{-1}(y)$. Thus K is a point in Z.

We now prove uniqueness. Suppose $f = q \circ p$ is another such factorization of f, where $p(X) = W$, and $q(W) = Y$. Let $w \in W$. Since p is monotone, $p^{-1}(w)$ is connected. If $y = q(w)$ then

$$p^{-1}(w) \subset p^{-1} \circ q^{-1}(y) = f^{-1}(y) \quad .$$

This implies that $p^{-1}(w)$ is contained in some component K of $f^{-1}(y)$. Also, $p(K)$ is connected and contained in $q^{-1}(y)$ which is totally disconnected. Therefore $p(K) = w$, which implies that $p^{-1}(w) = K$. From this it follows that W is homeomorphic to Z so that



commutes. □

If an additional assumption of differentiability is imposed upon mappings between spaces, it is not in general possible to factor them by means of differentiable mappings. Nor is it even possible to ensure that the intermediate space, Z, of Prop. 2.7 is a differential manifold. However, if more conditions are imposed upon f, such a factorization is possible.

Prop. 2.8 Let $f: M^n \to N^p$ be a C^m $(m \geq n)$ onto map which preserves

compactness under inverse images. Let $\sum(f)$ be the critical set of f. If dim $\sum(f) \leq n-2$, and dim $f(\sum(f)) \leq p-3$, then there is a factorization $f = h \circ g$ such that

(i) $g: M^n \to K^p$ is a C^m monotone map onto the C^m manifold K^p, and

(ii) $h: K^p \to N^p$ is a k-to-1 diffeo-covering map.

In addition the factorization is unique (replacing homeomorphism by C^m diffeomorphism in the definition of topological equivalence). The condition C^m can be replaced by C^∞ or real analytic.

Pf. Church and Timourian [3]. □

CHAPTER III

ANALYTIC MANIFOLDS, MAPPINGS AND VARIETIES

Def. 3.1 Let $\mathbb{H}^n$ denote the half plane $\{(x_1,\ldots,x_n) \in \mathbb{R}^n: x_1 \geq 0\}$. A topological space M is said to be an n dimensional (topological) manifold with boundary if it is separable, and if for every $x \in M$ there is some open neighborhood U of x such that U is homeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$. The set of all $x \in M$ where every such neighborhood U is homeomorphic to $\mathbb{H}^n$ is denoted by ∂M, and is called the boundary of M. The set of points $M-\partial M$ is denoted by Int M, and is called the interior of M.

Convention. Whenever a manifold M is written with a superscript as M^n, it is understood that the manifold is n dimensional. A manifold without boundary will be a manifold for which the boundary is the empty set. Henceforth, a manifold will be a manifold with or without boundary unless otherwise specified.

Def. 3.2 A function $f_m = (f_1,f_2,\ldots,f_n)$ defined on an open set $\Omega \subset \mathbb{R}^m$ and mapping into $\mathbb{R}^n$ is said to be (real) analytic, or C^ω, at $x \in \Omega$ if there is some open set U in Ω containing x such that each f_i can be written as a converging power series around x in U. If f is analytic at every point of Ω it is said to be analytic in Ω.

A function f defined on an open set Ω' of $\mathbb{H}^m$ to $\mathbb{R}^n$ is said to be analytic at $x \in \Omega$ if there is some open set U_1

containing x in $\mathbb{R}^n$, and an open set U containing x in Ω, and a function $f': U_1 \to \mathbb{R}^n$ analytic at x such that $f'|U_1 \cap U = f|U_1 \cap U$.

Def. 3.3 Let M be an n dimensional topological manifold. Let A be a collection of pairs (U,ϕ), where the sets U form an open covering of M, and ϕ is a homeomorphism of U onto an open set of $\mathbb{R}^n$ or $\mathbb{H}^n$. The collection A is said to be an (analytic) atlas on M if for all (U_1,ϕ_1) and (U_2,ϕ_2) of A such that $U_1 \cap U_2 \neq \phi$, $\phi_2 \circ \phi_1^{-1}|\phi_1(U_1)$ and $\phi_1 \circ \phi_2^{-1}|\phi_2(U_2)$ are analytic mappings on $\phi_1(U_1)$ and $\phi_2(U_2)$ respectively. An analytic atlas A on M is said to be saturated if there does not exist an analytic atlas A' on M which strictly contains A (i.e., A is a maximal atlas on M).

The manifold M, together with an analytic saturated atlas on it is called an analytic manifold.

Remark. If the analyticity is replaced by differentiability in the previous definition, M is called a differential manifold.

Let $x \in M$. A pair (U,ϕ) of A such that $x \in U$ is called a chart around x.

Note that if an atlas can be found on a manifold, Zorn's Lemma ensures that a maximal atlas, or a saturated atlas, can be found. Therefore Euclidean space $\mathbb{R}^n$ is an analytic (and differential) manifold. Let $A = \{(\mathbb{R}^n,i)\}$, where i is the identity mapping on $\mathbb{R}^n$. Then A is an atlas on $\mathbb{R}^n$ inducing the canonical analytic (differential) structure on $\mathbb{R}^n$. Similarly, $\mathbb{H}^n$ is also an analytic (differential) manifold.

Def. 3.4 A subset N of a topological manifold M will be called a submanifold of M if it is a topological manifold under the inherited topology from M.

Def. 3.5 Let M and N be analytic manifolds. A mapping $f: M \to N$ is said to be analytic at $x \in M$ if there is some chart (U,ϕ) in M around x and some chart (V,ψ) in N around $f(x)$ such that $\psi \circ f \circ \phi^{-1}|\phi(U)$ is analytic.

The mapping f is said to be differentiable if analyticity is replaced by differentiability in the above definition.

Prop. 3.6 Let M, N and P be manifolds, f and g analytic functions from M to $\mathbb{R}$, and h_1 and h_2 analytic functions from M to N and N to P respectively. Then $f + g$, fg and $h_2 \circ h_1$ are analytic functions.

Pf. See Dieudonné ([4]; Vol. III, p. 13). □

Def. 3.7 Let M be an analytic manifold. A set $V \subset M$ is called an analytic variety in M if, for every $x \in M$ there exists some open set $\mathcal{O}$ containing x such that $V \cap \mathcal{O} = \{y \in \mathcal{O}: f_1(y) = f_2(y_2) = \ldots = f_n(y) = 0\}$ for some analytic functions $f_1,\ldots,f_n: \mathcal{O} \to \mathbb{R}$.

Remark. Suppose M and N are analytic manifolds, and f is an analytic function from M to N. Then for all $y \in N$, $f^{-1}(y)$ is an analytic variety in M. To prove this, let (V,ψ) be a chart at y such that $\psi(y) = 0$. Then if $\psi \circ f|f^{-1}(V) = (g_1,g_2,\ldots,g_n)$, $f^{-1}(y) \cap f^{-1}(V) = \{x \in f^{-1}(V): g_1(x) = g_2(x) = \ldots = g_n(x) = 0\}$.

The following properties of analytic varieties can be shown:

Prop. 3.8 (i) In every variety V there is some variety $W \subset V$, which is closed and nowhere dense in V such that for all $x \in V-W$, $V-W$ is locally homeomorphic at x to $\mathbb{R}^n$, for some n depending on x. (If n is independent of x and $V-W$ is an n dimensional manifold, then V is said to be n dimensional.)

(ii) Finite unions of analytic varieties are analytic varieties Arbitrary intersections of analytic varieties are analytic varieties.

(iii) An analytic variety in a manifold without boundary is locally homeomorphic to the cone of a polyhedron with even Euler characteristic. In other words, if V is an analytic variety and $x \in V$, then there exists some neighborhood N of x in V and a polyhedron P such that N is homeomorphic to $C(P) = P \times I/(x,1)\sim(y,1)$.

(iv) An analytic variety can be triangulated locally finitely.

Pf. The proof of (i) can be found in Whitney [22]. Sullivan [20] proves (iii), and the proof of (iv) can be found in Lojasiewicz [11].

For (ii), let V_1 and V_2 be two analytic M varieties in an analytic manifold M. Let $x \in M$. Choose an open set $\mathcal{O}$ containing x in M such that $V_1 \cap \mathcal{O} = \{y \in \mathcal{O}: f_1(y) = \ldots = f_n(y) = 0\}$, and $V_2 \cap \mathcal{O} = \{y \in \mathcal{O}: g_1(y) = \ldots = g_p(y) = 0\}$ for some real valued analytic functions $f_1,\ldots,f_n$ and $g_1,\ldots,g_p$. Then $(V_1 \cup V_2) \cap \mathcal{O} = \{y \in \mathcal{O}: (f_1^2 + f_2^2 + \ldots + f_n^2)(g_1^2 + g_2^2 + \ldots + g_p^2)(y) = 0\}$. Therefore $V_1 \cup V_2$ is an analytic variety. In addition $(V_1 \cap V_2) \cap \mathcal{O} = \{y \in \mathcal{O}: f_1(y) = \ldots = f_n(y) = g_1(y) = \ldots = g_n(y) = 0\}$.

Narasimhan ([15]; p. 27) shows that analytic varieties in a manifold obey a descending chain condition. If $V_1 \supset V_2 \supset V_3 \supset \ldots$ is

a sequence of analytic varieties in a manifold, then there exists some number j such that $V_j = V_{j+1} = \ldots$. Let $\{V_\gamma : \gamma \in \Gamma\}$ be a family of analytic varieties. The descending chain condition implies that there exists varieties $V_{\gamma_1}, \ldots, V_{\gamma_k}$ such that $\bigcap_{\gamma\in\Gamma} V_\gamma = V_{\gamma_1} \cap \ldots \cap V_{\gamma_k}$. Therefore if each V_γ is an analytic variety in M, so is $\bigcap_{\gamma\in\Gamma} V_\gamma$. □

Lemma 3.9 Let V be a compact one dimensional variety in a manifold without boundary. Then V is a finite union of simple closed curves (henceforth called circles), meeting each other at most at a finite number of points.

Pf. Because V is compact, Prop. 3.8 (iv) implies that V has a finite triangulation. Let σ be any one-simplex in the triangulation of V. If e is an end point of a simplex, Prop. 3.8 (iii) implies that there is an even number of distinct simplices $\sigma_1, \ldots, \sigma_{2n}$ such that the closure of the star neighborhood of e $\overline{S(e)} = \sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_{2n}$. Choose any simplex σ of the simplices $\sigma_1, \ldots, \sigma_{2n}$, and let e be the end point of σ distinct from e'. Repeat the procedure above with e replaced by e' and choose similarly a simplex σ' from e' which is different from σ. In this way, a sequence of simplices $\sigma, \sigma', \sigma'', \ldots$ is built continuing until a point $e^{(n)}$ is found which is identical to some previous $e^{(m)}$. Then $\sigma^{(m)} \cup \sigma^{(m+1)} \cup \ldots \cup \sigma^{(n-1)}$ is a circle. Repeat the entire procedure to choose another circle, making sure that each simplex chosen at each end point is distinct from $\sigma^{(m)}, \ldots, \sigma^{(n-1)}$. This can be done befause an even number of simplices remain to be chosen at each end point.

The entire procedure can be repeated only a finite number of times, so that $V = C_1 \cup \ldots \cup C_n$ is a union of circles satisfying the conditions of the lemma. □

CHAPTER IV

COMPLETE REGULARITY AND 0-REGULARITY

Note: On a metric space X, the open ball of radius ε around a point $x \in X$ will be denoted by $B(x,\varepsilon)$. The closed ball of radius ε around x will be denoted by $\bar{B}(x,\varepsilon)$.

Def. 4.1 Let X and Y be metric spaces. An onto mapping $f: X \to Y$ is said to be proper if for every compact set $K \subset Y$, $f^{-1}(K)$ is compact.

A proper open mapping of X onto Y is said to be (homotopically) 0-regular at a point $x \in X$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all y, p and p' where $y \in Y$ and p, $p' \in f^{-1}(y) \cap B(x,\delta)$, p and p' are in the same arc-component (arcwise connected component) of $f^{-1}(y) \cap B(x,\varepsilon)$. If f is 0-regular at every $x \in X$, f is said to be 0-regular on X.

Def. 4.2 Let X and Y be metric spaces. A mapping $f: X \to Y$ is called completely regular if for all $y \in Y$ and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that there is a homeomorphism from $f^{-1}(y)$ to $f^{-1}(y')$ with the property that the distance between any point of $f^{-1}(y)$ and its image is less than ε for any $y' \in Y$ for which the distance between y and y' is less than δ. (Such a homeomorphism is called an ε-homeomorphism.)

Lemma 4.3 Let X and Y be metric spaces with metrics d_1, and d_2 respectively. Let $f: X \to Y$ be onto.

(a) If f is proper then for all $y \in Y$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in X$ and $d_2(y,f(x)) < \delta$, then $d_1(f^{-1}(y),x) < \varepsilon$.

(b) If f is open for all $y \in Y$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that if $f(x) = y$, and $d_2(y,y') < \delta$, then $d_1(x,f^{-1}(y')) < \varepsilon$.

Pf. (a) Suppose $y \in Y$ and $\varepsilon > 0$, and there exists a sequence of points (x_i) such that $(f(x_i)) \to y$ in Y and $d_1(f^{-1}(y),x_i) \geq \varepsilon$ for all i. The set $\{f(x_1),f(x_2),\ldots\} \cup \{y\}$ is compact in Y and therefore $[\bigcup_{i=1}^{\infty} f^{-1}(x_i)] \cup f^{-1}(y)$ is compact in X (since f is proper). Thus the set $\{x_i\}$ has a cluster point c in $[\bigcup_{i=1}^{\infty} f(x_i)] \cup f^{-1}(y)$. Then $f(c) = y$ which implies that $c \in f^{-1}(y)$ contradicting the fact that $d_1(f^{-1}(y),x_i) \geq \varepsilon$ for all i.

(b) Suppose there is some $y \in Y$ and $\varepsilon > 0$ such that for all $\delta > 0$ there is some $x \in f^{-1}(y)$ and $y' \in Y$ satisfying

$$d_2(y,y') < \delta \quad \text{and} \quad d_1(x,f^{-1}(y')) \geq \varepsilon .$$

Then there exists a sequence $(y_i) \to y$ and a sequence (x_i) in $f^{-1}(y)$ such that $d_1(x_i, f^{-1}(y_i)) \geq \varepsilon$. Because $f^{-1}(y)$ is compact, (x_i) has a cluster point c. Let $\mathcal{O}$ be an open set of X containing c and disjoint from each $f^{-1}(y_i)$. Then $y \in f(\mathcal{O})$ but $y_i \notin f(\mathcal{O})$ for any i. Therefore $f(\mathcal{O})$ is not open, which is a contradiction. □

The concepts of complete regularity and 0-regularity can be connected by the following proposition:

Prop. 4.4 Let X and Y be metric spaces. Suppose $f: X \to Y$ is a 0-regular mapping such that for all $y \in Y$, $f^{-1}(y)$ is a circle. Then

f is completely regular.

Pf. Let X and Y have metrices d_1 and d_2 respectively. Assume f is 0-regular. Then given $y \in Y$ and $\varepsilon > 0$, for all $x \in f^{-1}(y)$ there exists a $\gamma_x > 0$, such that if p and q are in $f^{-1}(y') \cap B(x,\gamma_x)$, then p and q are in the same arc component of $f^{-1}(y') \cap B(x,\varepsilon/2)$. By choosing γ_x sufficiently small for each x, γ_z can be chosen equal to γ_x for z sufficiently close to x in $f^{-1}(y)$. Therefore, because $f^{-1}(y)$ is compact, a γ can be chosen so that for all $x \in f^{-1}(y)$ if p and q are in $f^{-1}(y') \cap B(x,\gamma)$, then p and q are in the same arc component of $f^{-1}(y') \cap B(x,\varepsilon/2)$.

Let K be a triangulation of $f^{-1}(y)$ of mesh smaller than $\gamma/2$. By Lemma 4.3 there exists some $\delta > 0$ such ath if $d_2(y',y) < \delta$ and $x \in f^{-1}(y)$, then $d_1(x,f^{-1}(y')) < \frac{\gamma}{2}$. Choose any such y'.

For each of the finite set of points of the 0-skeleton of K, say e, choose a distinct point e' of $f^{-1}(y')$ such that $d_1(e,e') < \frac{\gamma}{2}$. The points, e', induce a triangulation of $f^{-1}(y')$, L, and form its 0-skeleton. Thus there is a 1-1 correspondence between the simplices of K and L. Extend the mapping of the 0-skeleton of K to L to a homeomorphism of $f^{-1}(y)$ to $f^{-1}(y')$ that carries simplices to simplices. Call this mapping g.

The homeomorphism g, is an ε-homeomorphism, for if e_1 is in the 0-skeleton of K, and is the end point of a 1-simplex (whose other end point is e_2), then $f(e_1)$ and $f(e_2)$ are in $f^{-1}(y') \cap B(e_1,\gamma)$, and therefore the simplex $g(\sigma)$ is in $B(e_1,\varepsilon/2)$, as is σ. Thus $g: \sigma \to g(\sigma)$ must be an ε-homeomorphism, which implies that $g: f^{-1}(y) \to f^{-1}(y')$ is an ε-homeomorphism. □

Def. 4.5 Let X be a metric space. The (Lebesgue) covering dimension of X, dim X is defined as follows:

(i) dim $X = -1$ iff $x = \emptyset$

(ii) dim $X \leq n$ if whenever $\mathcal{U}$ is a finite open cover of X, there exists some open refinement $\mathcal{V}$ of $\mathcal{U}$ covering X such that the maximum number of sets of $\mathcal{V}$ meeting each other at a point is at most $n+1$.

One says that dim $X = n$ whenever dim $X \leq n$ and it is not the case that dim $X \leq n-1$. If dim $X \neq n$ for all $n = -1, 0, 1, 2, \ldots$, then one says that dim $X = \infty$ (i.e., X is infinite dimensional).

Remark Nagata ([14]; p. 97) shows that the covering dimension of any topological n-dimensional manifold is n.

Prop. 4.6 Let X be a complete metric space, and Y a metrit space with finite covering dimension. Suppose $f: X \to Y$ is a completely regular mapping, and each point inverse of f is homeomorphic to a compact space M. Then $f: X \to Y$ is locally trivial. If Y is separable, locally compact, and contractible, then f is trivial.

Pf. Dyer and Hamstrom [5] proved this proposition (for all n) under the assumption that dim $Y \leq n+1$ and that the space of homeomorphisms of M to itself, $H(M)$, with the compact-open topology is locally n-connected. Cernavskii [2] showed that $H(M)$ is locally n-connected when M is compact by proving that $H(M)$ is locally contractible.

Steenrod ([19]; p. 53) shows that a locally trivial mapping $f: X \to Y$ is trivial when Y is locally compact, separable and contractible. □

CHAPTER V

THE TRIANGULATION AND CLASSIFICATION OF SURFACES

Def. 5.1 A compact, connected two-dimensional topological manifold with boundary will be called a surface.

Note: Henceforth, a closed arc in a space will be a homeomorph of the closed interval $[0,1] \subset \mathbb{R}$. An open arc will be a homeomorph of the open interval $(0,1) \subset \mathbb{R}$. Unless specified an arc will be closed.

Prop. 5.2 (Schoenflies Theorem) Let C be a circle in the sphere S^2. Then there exists a homeomorphism $h: S^2 \to S^2$ such that $h(C) = S^1 = \{(x,y,z) \in S^2 : z = 0\}$. Also, no arc of S^2 disconnects S^2.

Pf. See Moise ([12]; pp. 65-70).

Def. 5.3 Let M^2 be a surface. An open set $R \subset M^2$ is called a Jordan region if its closure in M^2 is homeomorphic to the disc D^2 in such a way that the boundary of R in M^2 is mapped onto ∂D^2.

Remark It is immediate from the Schoenflies Theorem that for any circle C in S^2, S^2-C is the union of two distinct Jordan regions. If C is a circle in $\mathbb{R}^2$, then the bounded component of $\mathbb{R}^2-C$ must also be a Jordan region in $\mathbb{R}^2$, because the one point compactification of $\mathbb{R}^2$ is homeomorphic to S^2.

Def. 5.4 Let J be a Jordan region with boundary γ in a surface.

A cross cut of J is the interior of an arc in the closure of J such that the end points of the arc, and only the end points, lie in the boundary γ.

Lemma 5.5 Let J be a Jordan region in a surface. If the boundary of J in the surface is γ, and σ is a cross cut of J, then σ divides J into two Jordan regions. The boundary of each Jordan region is σ, together with an arc of γ joining the end points of σ.

Pf. Let the two arcs of γ joining the end points of σ be γ_1 and γ_2. Then $\gamma_1 \cup \sigma$ and $\gamma_2 \cup \sigma$ are two circles which must, by Prop. 5.2, enclose Jordan regions J_1 and J_2. Because some boundary points of J_1 are in J and no boundary point of J is in J_1 it follows that $J_1 \subset J$, as similarly $J_2 \subset J$. The boundaries of J_1 and J_2 are not identical and neither Jordan region has a boundary point in the other region. Therefore they are disjoint.

To see that $J-\sigma$ is no more than the union of the two disjoint regions J_1 and J_2, identify the points of γ. In the quotient topology, the closure of J becomes a sphere and σ (with its boundary points) becomes a circle. Then Prop. 5.2 implies that $J-\sigma$ must be the two Jordan regions J_1 and J_2. □

In this chapter, the Schoenflies Theorem will be used to show that if C is any circle in a surface without boundary, M^2, then there is a triangulation of M^2 such that C is triangulated as a subpolyhedron of M^2. The proof of this is drawn from Ahlfors and Sario [1].

The first step towards the proof of this result is the following lemma.

Lemma 5.6 Let $\mathcal{U}$ be a finite covering of a surface without boundary, M^2, such that every set of $\mathcal{U}$ is a Jordan region and such that the intersection of the boundaries of any two regions of $\mathcal{U}$ is a finite set of arcs and points. Suppose furthermore that C is a circle in M^2 such that the intersection of C and a boundary of any region of $\mathcal{U}$ is also a finite set of arcs and points. In addition, assume C is not contained in any region of $\mathcal{U}$. Then M^2 may be triangulated so that C is a subpolyhedron of M^2.

Pf. Let $\mathcal{U}$ be the finite covering $\{J_i: i = 1,2,\ldots,n\}$. Assume without loss of generality that it is not the case that $J_i \subset J_j$ for $i \neq j$. For each J_i, let γ_i be its boundary in M^2.

Suppose there is some $\gamma_i \subset J_j$. Then since J_i is not contained in J_j, γ_i must enclose a Jordan region in J_j that is complementary to J_i. This implies that M^2 is a sphere. Prop. 5.2 implies that M^2 may be triangulated so that C is a subpolyhedron of M^2.

Assume that there is no $\gamma_i \subset J_j$. Lemma 5.5 implies that $M^2 - (\gamma_1 \cup \ldots \cup \gamma_n \cup C)$ is a finite pairwise disjoint union of Jordan regions. Let the region J_i be divided up by cross cuts into regions $J_{i1}, J_{i2}, \ldots, J_{im}$. In addition, each γ_i (and C) is divided by cross cuts into arcs γ_{ij} (and C_j). Each J_{ik} has a boundary composed of unions of such arcs.

To triangulate M^2, define a complex K whose 0-skeleton is the end points of the arcs γ_{ij} and C_j, together with an interior point of each γ_{ij}, C_j and each region J_{ik}. Join each such interior point, a, of J_{ik} to the o-skeleton points on the boundary of J_{ik} by means of arcs which are pairwise disjoint except at a. These arcs

divide J_{ik} into Jordan regions whose boundaries are formed by three arcs meeting at the points of the 0-skeleton. Thus the 1-simplices and the 2-simplices are induced naturally from the arcs and regions. □

Lemma 5.7 Let E be the union of a finite number of circles in a surface M^2. Let O be a connected open set in M^2. Then any two points of O may be connected by an arc whose intersection with E is the union of a finite number of arcs and points.

Pf. Choose any $p \in O$. Let H_p be the set of all points in which can be joined to p by the conditions of the lemma above. It will first be shown that O is open. Suppose p is not in E. Then there is some open connected set around p that is disjoint from E. Then this open set is contained in H_p. Suppose p is in E. Choose an open set of O around p which intersects only those circles of E to which p belongs. Suppose q is an element of this open set. There exists an arc γ joining q to p in O. Let C be the first point where γ meets E in passing from q to p. Define a new arc γ' which travels along γ from q to C and then travels along an arc of E until it meets p. Then γ' intersects E in a single arc.

It follows that H_p is open. However the relation between points of being joined by an arc meeting E in a finite number of points and arcs is an equivalence relation. Thus O is divided into open equivalence class by this method. Because O is connected, one of these classes must be all of O. Therefore $H_p = O$. □

Prop. 5.8 Let M^2 be a surface without boundary, and C a circle in M^2. Then there exists a triangulation of M^2 such that C is a sub-polyhedron of M^2.

Pf. By Lemma 5.6, it suffices to show that there exists an open cover of M^2 by Jordan regions such that the intersection of their boundaries with each other, or with C is a finite union of points and arcs.

Note first that the Jordan regions form a basis for the topology of M^2. Therefore two finite sets of Jordan regions $\{V_n\}$ and $\{W_n\}$ can be chosen so that $\bar{V}_n \subset W_n$, and $\{V_n\}$ covers M^2. Choose a sufficiently fine cover that C is not contained in any W_n. The Jordan regions described in Lemma 5.6 will be defined inductively. Let $J_1 = V_1$. Suppose $J_1,\ldots,J_{n-1}$ are defined. Represent $\bar{W}_n$ as a closed disc such that its centre lies in $V_n \subset W_n$. Choose two points on distinct radii of W_n so that p and q are not in C and the boundaries of $J_1,\ldots,J_{n-1}$, and so that the segments of the respective radii from p and q to the boundary of W_n are disjoint from $\bar{V}_n$. Extend these segments toward the centre of W_n until they meet $\bar{V}_n$, and call these extend segments σ_p and σ_q respectively.

The methods of Lemma 5.5 show that σ_p and σ_q divided $W_n - \bar{V}_n$ into two Jordan regions A and B. In each of A and B, join p and q by cross cuts τ_1 and τ_2 respectively, so that τ_1 and τ_2 meet C and the boundaries of $J_1,\ldots,J_{n-1}$ in a finite number of arcs and points (Lemma 5.7). Then $\tau_1 \cup \tau_2$ is a circle enclosing V_n in a Jordan region J_n. □

The following results will also be used in the next chapter:

Lemma 5.9 Let A^2 be an annulus with boundary. Let ℓ be an arc in A^2 joined at each end to different boundary components of A^2 and contained everywhere else in the interior of A^2. Then $A^2 - (\ell \cup \partial A^2)$ is an open disc.

Pf. Form a quotient space of A^2, $A^2/\sim$, by identifying any two points in the same compoent of ∂A^2. Then $A^2/\sim$ is a homeomorph of the sphere S^2. The arc ℓ becomes an arc ℓ' in S^2. Theorem 5.2 implies that $A^2/\sim - \ell'$ is an open disc. But $A^2/\sim - \ell'$ is homeomorphic to $A^2 - (\ell \cup \partial A^2)$. □

Def. 5.10 Let M^2 and N^2 be surfaces with boundary. Let ℓ_1 and ℓ_2 be two closed arcs in ∂M^2 and ∂N^2 respectively. Let $f: \ell_1 \to \ell_2$ be a homeomorphism. Then $M^2 \cup_f N^2$ is called a connected sum of M^2 and N^2.

Def. 5.11 Let M^2 and N^2 be surfaces without boundary. Let J_1 and J_2 be Jordan regions in M^2 and N^2 respectively. Then M^2-J_1 and N^2-J_2 are manifolds whose boundaries are circles. Let f be a homeomorphism from $\partial(M^2-J_1)$ to $\partial(M^2-J_2)$. Then $(M^2-J_1) \cup_f (N^2-J_2)$ is called a connected sum of M^2 and N^2.

Prop. 5.12 Let M^2 be a surface. If M^2 has no boundary, then M^2 is one of

(a) a sphere

(b) a finite connected sum of tori

(c) a connected sum of projective planes.

If M^2 has a non-empty boundary, then M^2 is a finite con-

nected sum of annuli, Moebius bands and handles (tori, each with a Jordan region removed). In all cases above, each representation is unique.

Pf. See Griffiths ([6]; pp. 58-61).

CHAPTER VI

THE CLASSIFICATION OF REAL VALUED, MONOTONE, C^ω FUNCTIONS

In this chapter, theorems will be developed to describe the real valued, monotone, C^ω functions on S^2, P^2 and T^2 up to topological equivalence.

Lemma 6.1 Let V be a compact 1-dimensional variety. Then there is a finite set $W \subset V$ such that $V-W$ is an analytic manifold.

Pf. By Prop. 3.8 (i) there is some compact analytic variety W such that $V-W$ is an analytic manifold, and W is nowhere dense in V. By Prop. 3.8 (iv), W has a finite triangulation, and therefore must be a finite set of points. □

Prop. 6.2 Let M^2 be a compact connected manifold with some compatible metric. Suppose f is a real valued C^ω function on M^2, say onto $[a,b]$. Suppose, also, that $f^{-1}(a) \cup f^{-1}(b)$ contains ∂M^2 and that $f^{-1}(y)$ is a circle for every $y \in (a,b)$. Then f is completely regular on $f^{-1}((a,b))$.

Pf. It will first be shown that f is 0-regular on $f^{-1}((a,b))$. It is clear that f is proper. Suppose J is a Jordan region in $f^{-1}((a,b))$. Then for every $y \in f(J)$, $f^{-1}(y)$ must disconnect J. Thus, by Prop. 2.1, y is not an extremum of the interval $f(J)$. So $f(J)$ is open. This implies that the mapping f is open on $f^{-1}((a,b))$.

Because 0-regularity at a point is a local property, and is

not a metric property, it can be assumed that Int M^2 is $\mathbb{R}^2$ to prove 0-regularity. Suppose $x \in \text{Int } M^2 = \mathbb{R}^2$. Let $F = f^{-1}f(x)$. It is possible to choose ε transversal to the variety F (which is an analytic manifold in a deleted neighborhood of x by Lemma 6.1). See Guillemin and Pollack ([7]; pp. 68-69) for this result about transersality. Because $F \cap \partial B(x,\varepsilon)$ is a nowhere dense compact polyhedron in $\partial B(x,\varepsilon)$ it must be a finite set of points, by Prop. 3.8 (iv). The transversality of $\partial B(x,\varepsilon)$ and F implies that the number of points in their intersection is twice the number of components of $F \cap \bar{B}(x,\varepsilon)$. For sufficiently small $\varepsilon > 0$, F is not contained in $B(x,\varepsilon)$, and therefore the component of $F \cap \bar{B}(x,\varepsilon)$ containing x, F_x, is an arc which meets $\partial B(x,\varepsilon)$ only at its end points, a and b. Because $\partial B(x,\varepsilon)$ and F are transversal at a and b, at each small arcs γ_a and γ_b can be chosen as neighborhoods of a and b and transversal to F.

Let r_a and r_b be analytic homeomorphisms on $[-1,1]$ onto γ_a and γ_b respectively, such that $r_a(0) = a$ and $r_b(0) = b$. Because $f \circ r_a$ and $f \circ r_b$ are analytic functions, they can fold (i.e., have relative maxima and minima) at only a finite set of points in $[-1,1]$. Therefore f can fold only at a finite set of points in γ_a and γ_b. Choose γ_a and γ_b sufficiently small that a and b are the only possible points in these arcs at which f folds. If f did fold at a or b, then F would not disconnect M^2. This cannot be, and therefore f, restricted to each arc, γ_a and γ_b, is a homeomorphism. Choose γ_a and γ_b such that $f(\gamma_a) = f(\gamma_b)$.

Let $\Gamma = \bar{B}(x,\varepsilon) \cap f^{-1}f(x)$ and $S = [\Gamma \cap \partial B(x,\varepsilon)] - (\gamma_a \cup \gamma_b)$. Let the distance between F_x and S be $\ell > 0$. Let λ be the minimum of the distances between F_x and the other components of $F \cap \bar{B}(x,\varepsilon)$.

By Lemma 4.3 (a), for γ_a and γ_b sufficiently small, the distance between any point of Γ and $F \cap \bar{B}(x,\varepsilon)$ will be less than $\beta = \min(\frac{\lambda}{2}, \ell)$. Let P be the component of some such preimage fiber $F' \cap \bar{B}(x,\varepsilon)$ which meets γ_a. For each component of $F \cap \bar{B}(x,\varepsilon)$, the set of points of P closer than β to that component is open in P. Because P is connected, it follows that one such set is all of P. However, this set must be close to F_x if γ_a is sufficiently small. This implies that P contains no point of S, and therefore must be attached at its other end to γ_b. The set of such fiber components P meeting γ_a is a neighborhood of x, and therefore contains some ball $B(x,\delta)$. Thus 0-regularity holds at x.

Complete regularity now follows from Prop. 4.4. □

Prop. 6.3 Let D^2 be a closed disc. Suppose that there exists a finite set $W \subset \partial D^2$ such that $D^2 - W$ is an analytic manifold. Let f be a monotone real valued function on D^2 such that f is C^ω on $D^2 - W$, and f takes an extreme value precisely on ∂D^2. Then f is topologically equivalent to the function $g(x,y) = x^2 + y^2$ on $B^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Pf. Assume without loss of generality that $f(D^2) = [0,1]$, and $f^{-1}(1) = \partial D^2$. Suppose $y \in (0,1)$. Then $f^{-1}(y)$ disconnects D^2 into two components. The set $f^{-1}(y)$ cannot contain an open set that is homeomorphic to $\mathbb{R}^2$ because it would then contain an open set of D^2. This would imply that f is constant on D^2, which is not the case. If $f^{-1}(y)$ were a connected 0-dimensional variety, Prop. 3.8 (iv) would imply that it is a single point, which could not disconnect D^2.

Thus, since $f^{-1}(y)$ is connected, Lemma 3.9 implies that it is a finite connected union of circles. However, if $f^{-1}(y)$ were a connected union of more than one circle, then by Prop. 5.2, $f^{-1}(y)$ would disconnect D^2 into more than two components. By Prop. 2.1, this cannot be, and therefore $f^{-1}(y)$ is a circle. So $\{f^{-1}(y): y \in (0,1]\}$ must be a collection of nested circles. As before, $f^{-1}(0)$ cannot contain an open set homeomorphic to $\mathbb{R}^2$. If $f^{-1}(0)$ were 1-dimensional, it would be a finite connected union of circles, and therefore $D^2-f^{-1}(0)$ would be disconnected, which is not the case. Therefore $f^{-1}(0)$ must be a point.

The 0-regularity of f on $f^{-1}((0,1))$ follows as in Prop. 6.2. On ∂D^2, a slight modification of the proof is required. For any $x \in \partial D^2$, x has a deleted neighborhood in which ∂D^2 is an analytic manifold. The proof of 0-regularity carries through as in Prop. 6.2 except that γ_a and γ_b are chosen to be one sided neighborhoods of a and b respectively in $\partial B(x,\varepsilon)$. Complete regularity and triviality on $f^{-1}((0,1])$ follow from Propositions 4.4 and 4.6. So f must be topologically equivalent on $f^{-1}((0,12)$ to the projection mapping $\pi_2: S^1 \times (0,1] \to (0,1]$, as is g. The transitivity of the equivalence implies that f is equivalent to g on $D^2-f^{-1}(0)$ and $B^2-(0,0)$ respectively. Therefore, there exist two homeomorphisms α and β such that the diagram

$$\begin{array}{ccc} D^2-f^{-1}(0) & \xrightarrow{\alpha} & \{B^2-(0,0)\} \\ \Big\downarrow f & & \Big\downarrow g \\ (0,1] & \xrightarrow[\beta]{} & (0,1] \end{array}$$

commutes. To establish the equivalence of f and g, let $\alpha(f^{-1}(0)) = (0,0)$ and $\beta(0) = 0$. □

Lemma 6.4 Let D_1^2 and D_2^2 be two discs, and let f_1 and f_2 be two real valued functions on D_1^2 and D_2^2 respectively such that both are topologically equivalent to $g: B^2 \to \mathbb{R}$ of Prop. 6.3. Suppose α is a homeomorphism from ∂D_1^2 to ∂D_2^2. Then α can be extended to a homeomorphism α^* from D_1^2 to D_2^2 such that there is some $\beta: \mathbb{R} \to \mathbb{R}$ making

$$\begin{array}{ccc} D_1^2 & \xrightarrow{\alpha^*} & D_2^2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \mathbb{R} & \xrightarrow[\beta]{} & \mathbb{R} \end{array}$$

commute.

Pf. It can be assumed without loss of generality that $D_1^2 = D_2^2 = B^2$, and $f_1 = f_2 = g$.

Define $\alpha^*(s,t) = \left[\alpha\left(\frac{s}{\sqrt{s^2+t^2}}, \frac{t}{\sqrt{s^2+t^2}}\right)\right]\sqrt{s^2+t^2}$

$$\alpha^*(0,0) = (0,0)$$

and define β to be the identity mapping on $\mathbb{R}$. Henceforth, α^* will be denoted by α. □

Def. 6.5 Let M^{n-1} be a submanifold of N^n. Then M^{n-1} is said to be two sided in N^n if M^{n-1} is connected and has an open connected neighborhood W such that $W - M^{n-1}$ has two components, U and V,

each open in N^n and such that the boundary of each in W is M^{n-1}. The manifold M^{n-1} is said to be locally two sided at $x \in M^{n-1}$ if there exists an open neighborhood $\mathcal{O}$ of x such that $M^{n-1} \cap \mathcal{O}$ is connected and two sided in $\mathcal{O}$. In this case, the closures of U and V in $\mathcal{O}$ are called one sided neighborhoods of x.

Remark. Let M^1 be a submanifold of a manifold N^2 such that $M^1 \cap \partial N^2 = \partial M^1$. Then M^1 is locally two sided in N^2.

To see this, suppose $x \in \text{Int } M^1$. Choose a Jordan region J containing x, and let M_x be the component of $M^1 \cap J$ containing x. By Lemma 5.5, M_x divides J into two Jordan regions. Thus M^1 is two sided at x. Suppose $x \in \partial M^1$. Reduce this situation to the case where $N^2 = \mathbb{H}^2$ and $x = (0,0)$. Repeat the procedure above, using the half disc $D_+ = \{(x,y) \in \mathbb{H}^2 : x^2+y^2 < 1\}$. In this case, $D_+ - \partial N^2$ is a Jordan region.

Let M^2 be the Moebius band, and C the central circle of M^2. Although C is locally two sided in M^2, it is not two sided in M^2.

Def. 6.6 Let M^1 be a closed arc or a circle in a surface N^2 such that $M^1 \cap \partial N^2 = \partial M^1$. Let ($N^2$ cut M^1) be the set $N^2 - M^1$ together with two copies, $(p,0)$ and $(p,1)$ of every point $p \in M^1$.

Define a topology on (N^2 cut M^1) in the following fashion. If $x \in N^2 - M^1$, choose a neighborhood base at x for (N^2 cut M^1) of neighborhoods of x in the topology of N^2-M^1. If M^1 is a closed arc in N^2, then by Rushing [18], M^1 is two sided in N^2. Choose a one sided neighborhood U_o of M^1 (i.e., the closure in W of one

of U or V from Def. 6.5). The neighborhoods of points of the form $(p,0)$ will be neighborhoods of p in U_o with points $q \in M^1$ replaced by points $(q,0)$. The neighborhoods of $(p,1)$ are defined similarly, using the neighborhood U_1 on the other side of M^1. If M^1 is a circle, choose $c \in M^1$. Then $M^1 - \{c\}$ is an arc which is two sided in N^2 as before. Neighborhoods of points (p,i), $i = 0$ or 1, $p \in M^1 - \{c\}$ can be chosen as above. To define open neighborhoods of $(c,0)$ and $(c,1)$, choose a one sided neighborhood base at c. An open neighborhood of $(c,0)$ will be a one sided open neighborhood A from this base, where $p \in M^1 \cap A$ is replaced by $(p,0)$ if $(A \cap U_o) - M^1$ has p as a boundary point. The point p is replaced by $(p,1)$ if $(A \cap U_1) - M^1$ has p as a boundary point. For $(c,1)$, a one sided neighborhood B on the other side of M^1 is used instead of A. The space $(N^2$ cut $M^1)$ is a 2-dimensional manifold with boundary.

Note. In the previous definition, suppose a point $d \in M^1$, a circle, is chosen instead of $c \in M^1$. Let $(N^2$ cut $M^1)_c$ be $(N^2$ cut $M^1)$ defined using point c, and $(N^2$ cut $M^1)_d$ be $(N^2$ cut $M^1)$ using point d. There is a homeomorphism of the two copies of M^1 in $(N^2$ cut $M^1)_c$ onto the two copies of M^1 in $(N^2$ cut $M^1)_d$. Let (p,i) be a copy of a point p of M^1 in $(N^2$ cut $M^1)_c$. The neighborhoods of (p,i) are half neighborhoods in N^2 of p with appropriate substitutions of points q of M^1 by points of the form (q,j). Map (p,i) to $(p,0)$ or $(p,1)$ in $(N^2$ cut $M^1)_d$ so that the neighborhoods of the image are on the same side of M^1 as those of (p,i) in $(N^2$ cut $M^1)_c$. Extend this mapping to all of $(N^2$ cut $M^1)_c$

by defining it to be the identity mapping on N^2-M^1. Then $(N^2 \text{ cut } M^1)_c$ is seen to be homeomorphic to $(N^2 \text{ cut } M^1)_d$, which shows that $(N^2 \text{ cut } M^1)$ is well defined up to the notion of homeomorphism.

Lemma 6.7 Suppose that there is some finite set $W \subset \partial N^2$ such that N^2-W is an analytic manifold. Suppose, also, that M^1 is a circle or an arc in N^2 such that $M^1 \cap \partial N^2 = \partial M^1$, and such that M^1 is an analytic manifold in N^2 except at a finite number of points. Then there exists some finite set $V \subset \partial(N^2 \text{ cut } M^1)$ such that $(N^2 \text{ cut } M^1)-V$ has an analytic structure compatible with the structure on $N^2-(M^1 \cup W)$.

Pf. Let F be the set of points of M^1 where M^1 fails to be an analytic manifold. Let G be the set of points which are copies in $(N^2 \text{ cut } M^1)$ of points of F together with the copies of the boundary points of M^1 (if any). Now define $V = (W-M^1) \cup G$.

The set $N^2 - (M^1 \cup W)$ is contained in $(N^2 \text{ cut } M^1)-V$, and has an analytic structure as an open subset of N^2-W. Consider any $x \in M^1$ which is not in ∂M^1 or in F. Then because M^1 is an analytic submanifold of N^2 at x, the one sided neighborhoods at x are analytically homeomorphic to neighborhoods of the origin in $\mathbb{H}^2$. Thus these analytic homeomorphisms, considered as one sided charts, define an analytic structure at the copies of points of M^1 in $(N^2 \text{ cut } M^1)$. □

Convention. Let $Q: (N^2 \text{ cut } M^1) \to N^2$ be the quotient mapping such that $Q(p,0) = Q(p,1) = p$ and such that Q is the identity on $N^2 - M^1$.

Suppose $A \subset N^2$. If the context is clear, for convenience the set $Q^{-1}(A)$ in $(N^2 \text{ cut } M^1)$ will be denoted simply as A. In a similar fashion, if $B \subset (N^2 \text{ cut } M^1)$, $Q(B)$ will be denoted by B. In addition, if f is a function on N^2, then $f \circ Q$, defined on $(N^2 \text{ cut } M^1)$ will be denoted by f. If f is a function on $(N^2 \text{ cut } M^1)$, and $f \circ Q^{-1}$ is a well defined function, then it, too, will be denoted by f.

Lemma 6.8 Let M^2 be a surface without boundary, and let C be a circle in M^2. Then the Euler characteristic $\chi(M^2 \text{ cut } C) = \chi(M^2)$.

Pf. By Prop. 5.8, simplicial complexes K and L can be chosen such that $|K| = C$, $|L| = M^2$, and K is a subcomplex of L.

Let n_i be the number of i-dimensional simplices in L. Then $\chi(M^2) = n_0 - n_1 + n_2$. If m_i is the number of i-dimensional simplices in K, then $\chi(C) = m_0 - m_1 = 0$. The triangulation of C induces a triangulation of $\partial(M^2 \text{ cut } C)$ by means of the inverse of the quotient map Q^{-1}. If s is a simplex of K, then $Q^{-1}(s)$ is the disjoint union of two simplices. Therefore, $\chi(M^2 \text{ cut } C) = (n_0+m_0) - (n_1+m_1) + n_2 = \chi(M^2)$. □

Lemma 6.9 Let C be a circle embedded in T^2 or P^2. If C does not disconnect T^2, then $(T^2 \text{ cut } C)$ is an annulus. If C does not disconnect P^2, then $(P^2 \text{ cut } C)$ is a disc. If C disconnects T^2, then $(T^2 \text{ cut } C)$ is the union of a disc and a torus with a hole. If C disconnects P^2, then $(P^2 \text{ cut } C)$ is the union of a disc and a Moebius band.

Pf. From Lemma 6.8, $\chi(T^2 \text{ cut } C) = \chi(T^2) = 0$. If C does not disconnect T^2, then $(T^2 \text{ cut } C)$ is a surface with boundary, which can be described by means of a plan $\zeta_{p,q,r}$, where p is the number of annuli, q is the number of Moebius bands, and r is the number of handles connected to form $(T^2 \text{ cut } C)$, by Prop. 5.12. But a surface S of plan $\zeta_{p,q,r}$ has characteristic $\chi(S) = 1-p-q-2r$. Therefore $(T^2 \text{ cut } C)$ must be of plan $\zeta_{0,1,0}$ or $\zeta_{1,0,0}$. These are, respectively, an annulus and a Moebius band. But $(T^2 \text{ cut } C)$ must be orientable because T^2-C is orientable, and therefore it must be an annulus.

If C does not disconnect P^2, then $(P^2 \text{ cut } C)$ is also a surface and $\chi(P^2 \text{ cut } C) = \chi(P^2) = 1$. Then if $(P^2 \text{ cut } C)$ has plan $\zeta_{p,q,r}$, it follows that $p = q = r = 0$. Therefore $(P^2 \text{ cut } C)$ is a disc.

If C disconnects T^2, then $(T^2 \text{ cut } C)$ must be the union of two surfaces. The characteristics of each cannot be 0 because in that case, both surfaces would be annuli. However, the boundary of each must be a circle. Therefore the characteristics must be -1 and 1. It follows that one surface is a disc and the other is of plan $\zeta_{1,1,0}$, $\zeta_{2,0,0}$, $\zeta_{0,2,0}$, or $\zeta_{0,0,1}$. But the only one of these that is orientable and that possesses a single boundary component is $\zeta_{0,0,1}$, a torus with a hole.

Finally, if C disconnects P^2, then the only possible plans for the two surfaces are $\zeta_{0,0,0}$ and $\zeta_{0,1,0}$ because both surfaces must have a single boundary component. Therefore, one surface is a disc, and the other is a Moebius band. □

Theorem 6.10 There exist precisely two topologically distinct real valued, monotone, C^ω functions on S^2.

Pf. One such class of functions is clearly the class of constant functions.

Assume that f is a nonconstant, monotone, real valued, C^ω function on S^2. Also assume, without loss of generality, that f maps S^2 onto $[-1,1]$. Then as in Prop. 6.3, $f^{-1}(y)$ must be a circle for every $y \in (-1,1)$. By Propositions 6.2 and 4.6, f is trivial on $f^{-1}((-1,1))$.

Suppose j is another such function that maps S^2 onto $[-1,1]$. Then j must be trivial on $j^{-1}((-1,1))$. Therefore there must exist homeomorphisms $\alpha: f^{-1}((-1,1)) \to j^{-1}((-1,1))$ and $\beta: (-1,1) \to (-1,1)$ so that

$$\begin{array}{ccc} f^{-1}((-1,1)) & \xrightarrow{\alpha} & j^{-1}((-1,1)) \\ \downarrow f & & \downarrow j \\ (-1,1) & \xrightarrow[\beta]{} & (-1,1) \end{array}$$

commutes. We extend α and β by defining $\beta(1) = \lim_{x \to 1} \beta(x)$, $\beta(-1) = \lim_{x \to -1} \beta(x)$, and $\alpha(f^{-1}(1)) = j^{-1}(\beta(1))$, $\alpha(f^{-1}(-1)) = j^{-1}(\beta(-1))$. Then

$$\begin{array}{ccc} S^2 & \xrightarrow{\alpha} & S^2 \\ \downarrow f & & \downarrow j \\ [-1,1] & \xrightarrow[\beta]{} & [-1,1] \end{array}$$

commutes. □

Theorem 6.11 There are exactly four topologically distinct montone, real valued, C^∞ functionons on the projective plane.

Pf. It is clear that the constant functions form one of these categories.

Suppose the inverse image of every value of f, a real valued, monotone, C^∞ function on P^2, is either a circle or a point. Then for all $y \in (0,1)$, $f^{-1}(y)$ is a circle which bounds in P^2. By Propositions 6.2 and 4.4.6, f is trivial on $f^{-1}((0,1))$. If $f^{-1}(0)$ and $f^{-1}(1)$ were both points then P^2 would be homeomorphic to S^2, which cannot be. Therefore assume that $f^{-1}(1)$ is a circle. By Lemma 6.9, $(P^2$ cut $f^{-1}(1))$ is a disc. Lemmas 6.1 and 6.7 imply that $(P^2$ cut $f^{-1}(1))$ is analytic except at a finite number of points on the boundary.

Suppose $j: P^2 \to\to [0,1]$ has the properties mentioned above for f. Then $(P^2$ cut $j^{-1}(1))$ is also a disc that is analytic except at a finite number of points on the boundary. In $\partial(P^2$ cut $f^{-1}(1))$, there are two copies of every point of $f^{-1}(1)$, and similarly in $\partial(P^2$ cut $j^{-1}(1))$ there are two copies of every point of $j^{-1}(1)$. Thus $Q_f: \partial(P^2$ cut $f^{-1}(1)) \to f^{-1}(1)$ and $Q_j: \partial(P^2$ cut $j^{-1}(1)) \to j^{-1}(1)$, the quotient mappings, are two-fold coverings of $f^{-1}(1)$ and $j^{-1}(1)$ respectively, and copies of points in $f^{-1}(1)$ and $j^{-1}(1)$ are the fibers. Choose a bundle isomorphism (α,γ), $\alpha: f^{-1}(1) \to j^{-1}(1)$ and $\gamma: \partial(P^2$ cut $f^{-1}(1)) \to \partial(P^2$ cut $j^{-1}(1))$. By Lemma 6.4, γ can be extended to a homeomorphism on all of $\partial(P^2$ cut $f^{-1}(1))$ so that there is some homeomorphism $\beta: \mathbb{R} \to \mathbb{R}$ making

$$\begin{array}{ccc} (P^2 \text{ cut } f^{-1}(1)) & \xrightarrow{\alpha} & (P^2 \text{ cut } j^{-1}(1)) \\ \downarrow f & & \downarrow j \\ \mathbb{R} & \xrightarrow[\beta]{} & \mathbb{R} \end{array}$$

commute. Extend α by defining it to be equal to γ on $P^2 - f^{-1}(1)$. Then

$$\begin{array}{ccc} P^2 & \xrightarrow{\alpha} & P^2 \\ \downarrow f & & \downarrow j \\ \mathbb{R} & \xrightarrow[\beta]{} & \mathbb{R} \end{array}$$

commutes. An example of this type of function is given in P2 of the Appendix.

Suppose that there is one $y \in [0,1]$ such that $f^{-1}(y)$ is not a circle or a point. If $f^{-1}(0)$ contained two distinct circles, C_1 and C_2, then $(P^2 \text{ cut } C_1)$ would be a disc, and C_2 would be a circle in $(P^2 \text{ cut } C_1)$ or an arc with ends attached to $\partial(P^2 \text{ cut } C_1)$. Thus $f^{-1}(0)$ would disconnect P^2 into two components, contradicting Prop. 2.1. Therefore $f^{-1}(0)$ must be a circle or a point. Similarly, $f^{-1}(1)$ must be a circle or a point. So $y \in (0,1)$. Let C_1 and C_2 be two circles, meeting at a finite number of points, such that $C_1 \cup C_2 \subset f^{-1}(y)$. At least one of them does not bound in P^2 because $P^2 - f^{-1}(y)$ has two components. Suppose this is C_1. Then $(P^2 \text{ cut } C_1)$ is a disc, by Lemma 6.9. Clearly there can only be one arc of $f^{-1}(y)$ in Int $(P^2 \text{ cut } C_1)$ in order to disconnect it into exactly two components. This means that $C_1 \cup C_2 = f^{-1}(y)$. There are two possibilities.

(i) In $(P^2 \text{ cut } C_1)$, C_2 is an arc attached at each end to C_1.

These end points must be in the same fiber of Q_f because C_2 is a circle in P^2. In this case, C_2 does not bound in P^2. An example is given in P3 of the Appendix.

(ii) In $(P^2 \text{ cut } C_1)$, C_2 is a circle that is attached to C_1 at one point. In this case, C_2 bounds in P^2. An example is given in P4 of the Appendix.

Suppose, in case (i), that j is another function with the properties described above for f. Let the preimage of some value of j be the union of two circles C_1' and C_2' such that neither C_1' nor C_2' bounds in P^2. Let C_2 meet C_1 in $(P^2 \text{ cut } C_1)$ at points a and b, and let C_2' meet C_1' in $(P^2 \text{ cut } C_1')$ at points a' and b'. Then a and b break C_1 into two arcs, M and N, meeting at a and b. Similarly, C_1' is broken into arcs M' and N'. Choose a bundle isomorphism (α,γ), $\alpha: C_1 \to C_1'$, $\alpha: \partial(P^2 \text{ cut } C_1) \to \partial(P^2 \text{ cut } C_1')$ such that $\gamma(a) = a'$ and $\gamma(b) = b'$. Because C_2 and C_2' are arcs in $(P^2 \text{ cut } C_1)$ and $(P^2 \text{ cut } C_1')$, γ can be extended to a homeomorphism from C_2 to C_2'. The sets $M \cup C_2$, $N \cup C_2$, $M' \cup C_2$ and $N' \cup C_2'$ enclose discs, and so by Lemmas 6.1, 6.7 and 6.4, γ can be extended to a homeomorphism from $(P^2 \text{ cut } C_1)$ to $(P^2 \text{ cut } C_1')$ such that there exists a homeomorphism $\beta: \mathbb{R} \to \mathbb{R}$ making

$$\begin{array}{ccc} (P^2 \text{ cut } C_1) & \xrightarrow{\gamma} & (P^2 \text{ cut } C_1') \\ \downarrow f & & \downarrow j \\ \mathbb{R} & \xrightarrow[\beta]{} & \mathbb{R} \end{array}$$

commute. Extending α as before

$$\begin{array}{ccc} P^2 & \xrightarrow{\alpha} & P^2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbb{R} & \longrightarrow & \mathbb{R} \end{array}$$

commutes.

In case (ii), the procedure is similar except that C_2 and C_2' are circles in $(P^2 \text{ cut } C_1)$ and $(P^2 \text{ cut } C_1')$ respectively. The bundle isomorphism (α,γ) is defined on $\partial(P^2 \text{ cut } C_1)$ as before. If D and D' are the two discs enclosed by C_2 and C_2' in $(P^2 \text{ cut } C_1)$ and $(P^2 \text{ cut } C_1')$ respectively, then $(P^2 \text{ cut } C_1) - \text{Int } D$ and $(P^2 \text{ cut } C_1') - \text{Int } D'$ are discs with two points on their boundaries identified. Let D_1 and D_1' be these two discs and let d, e and d', e' be the points which are identified by quotient maps, p and q. The mapping $q^{-1} \circ \gamma \circ p$ is a homeomorphism from the arc $p^{-1}(C_1) - \{d,e\}$ to the arc $q^{-1}(C_1') - \{d', e'\}$. Extending this to a homeomorphism $h: \partial D_1 \to \partial D_1'$ defines a homeomorphism $\gamma: \partial(P^2 \text{ cut } C_1) \cup C_2$ to $(P^2 \text{ cut } C_1') \cup C_2'$. We extend h to a homeomorphism $h: D_1 \to D_1'$ such that a homeomorphism $\beta: (f \circ p)(D_1) \to (j \circ q)(D_1')$ exists making

$$\begin{array}{ccc} D_1 & \xrightarrow{h} & D_1' \\ {\scriptstyle f \circ p}\downarrow & & \downarrow{\scriptstyle j \circ q} \\ (f \circ p)(D_1) & \xrightarrow[\beta]{} & (j \circ q)(D_1') \end{array}$$

commute (Lemma 6.4). If γ is defined to be equal to h on $(P^1 \text{ cut } C_1)-D$, then

$$\begin{array}{ccc} (P^2 \text{ cut } C_1)\text{-Int } D & \xrightarrow{\gamma} & (P^2 \text{ cut } C_1')\text{-Int } D' \\ \Big\downarrow f & & \Big\downarrow j \\ f[(P^2 \text{ cut } C_1)\text{-Int } D] & \xrightarrow[\beta]{} & j[(P^2 \text{ cut } C_1')\text{-Int } D'] \end{array}$$

commutes. The mapping γ can then be extended to Int D so that β has an extension to a homeomorphism on all of $\mathbb{R}$ making

$$\begin{array}{ccc} (P^2 \text{ cut } C_1) & \xrightarrow{\gamma} & (P^2 \text{ cut } C_1') \\ \Big\downarrow f & & \Big\downarrow j \\ \mathbb{R} & \xrightarrow[\beta]{} & \mathbb{R} \end{array}$$

commute. Then γ defines an extension of α as in case (i) so that

$$\begin{array}{ccc} P^2 & \xrightarrow{\alpha} & P^2 \\ \Big\downarrow f & & \Big\downarrow j \\ \mathbb{R} & \xrightarrow[\beta]{} & \mathbb{R} \end{array}$$

commutes. □

Lemma 6.12 Let f be a nonconstant C^ω function from T^2 to $\mathbb{R}$. Then there exists a $y \in f(T^2)$ such that $f^{-1}(y)$ is neither a point nor a circle.

Pf. Suppose this were not so. Without loss of generality, let $f(T^2) = [0,1]$. Then for all $y \in (0,1)$, $f^{-1}(y)$ is a circle. By Prop. 6.2, and Prop. 4.6, f is trivial on $f^{-1}((0,1))$. If both $f^{-1}(0)$ and $f^{-1}(1)$ were points, then T^2 would be homeomorphic to S^2, which cannot be. So either $f^{-1}(0)$ or $f^{-1}(1)$, say $f^{-1}(1)$, is

a circle. By Lemma 6.9, $(T^2 \text{ cut } f^{-1}(1))$ is an annulus. Thus $f: (T^2 \text{ cut } f^{-1}(1)) \to [0,1]$ has the property that $f^{-1}(1) = \partial(T^2 \text{ cut } f^{-1}(1))$ has two components. By Lemma 4.3 (a), for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in (T^2 \text{ cut } f^{-1}(1))$, if $f(x) \in (1-\delta, 1]$, then the distance between x and $\partial(T^2 \text{ cut } f^{-1}(1))$ is less than ε. If ε is chosen less than half the distance between the two components of $\partial(T^2 \text{ cut } f^{-1}(1))$, the sets $f^{-1}(y)$, where $y \in (1-\delta, 1)$ will be disconnected, which is a contradiction. □

Theorem 6.13 There are at most nine topologically distinct monotone, real valued, C^ω functions on a torus.

Pf. One class of functions that can be stated occurs when

1. The monotone C^ω real valued function is constant.

Henceforth, only the nonconstant functions will be considered. Let f be such a function on T^2. Assume without loss of generality that $f(T^2) = [0,1]$. Then by Lemmas 6.11 and 3.9, there exists a $y \in [0,1]$ such that $f^{-1}(y)$ is the finite connected union of two or more circles, connected at a finite number of points. Because $T^2 - f^{-1}(y)$ has one or two components, it follows that there is a circle $C \subset f^{-1}(y)$ such that $T^2 - C$ is connected. So by Lemma 6.9, $(T^2 \text{ cut } C)$ is an annulus. Let the remaining circles of $f^{-1}(y)$ be denoted by $C_1, C_2, \ldots, C_n$. Then in $(T^2 \text{ cut } C)$, $C_1 \cup C_2 \cup \ldots \cup C_n$ becomes a union of arcs, circles and discrete points on the boundary of $(T^2 \text{ cut } C)$. These discrete points are copies of points of arcs and circles that touch the boundary. Because they play no role in the subsequent analysis, these discrete points can be ignored. They are

determined totally by the placement of the arcs and circles.

Suppose $y = 0$ or $y = 1$, say 1. If any of $C_1, \ldots, C_n$ were circles in (T^2 cut C), then $T^2 - f^{-1}(1)$ would have at least two components. This is not the case, and therefore the arcs and circles $C_1, \ldots, C_n$ reduce to the case of a single arc, say C_1, attached at each end to a different component of $\partial(T^2$ cut $C)$. In other words, $f^{-1}(1) = C \cup C_1$, and $((T^2$ cut $C)$ cut $C_1)$ is a disc, by Lemma 5.9. So f must be equivalent to the function g of Prop. 6.3. Note that in (T^2 cut C), C_1 must be an arc attached at each end to points which are copies of each other. If this were not the case, then in T^2, $f^{-1}(1)$ would have points at which an odd number of branches of $f^{-1}(1)$ would meet. It may be concluded that $f^{-1}(1)$ is the point union of two circles in T^2, neither of which disconnects T^2.

The proof that these properties characterize f topologically proceeds in much the fashion of the case for P^2 represented by example P2 of the Appendix. Label points as in diagram 6.A.2. If j is another such function with corresponding circles C', C_1', and points a', b', c' and d', then a homeomorphism $\gamma: \partial((T^2$ cut $C)$ cut $C_1) \to \partial((T^2$ cut $C')$ cut $C_1')$ is chosen so that $\gamma(a) = a'$, $\gamma(b) = b'$, $\gamma(c) = c'$ and $\gamma(d) = d'$, and so that γ carries copies of points to copies of points. The proof then follows as before.

The remaining examples occur when any point y, such that $f^{-1}(y)$ is not a circle or a point, is not an extreme value of f. Assume $f(T^2) = [0,1]$ and that $y \in (0,1)$. In this case, the arcs $C_1, \ldots, C_n$ must disconnect (T^2 cut C) into two components. Suppose that none of these arcs and circles is attached to both boundary components. Then the closure of one component of $T^2 - (C \cup C_1 \cup \ldots \cup C_n)$

must be an annulus, and f is monotone and analytic on the interior of this annulus. Moreover, f attains one value precisely on its boundary. As in Lemma 6.12, this cannot be. Therefore there is an arc, L, which is attached at each end to a different component of $\partial(T^2$ cut C). Thus it follows that $((T^2$ cut C) cut L) is a disc.

There are two ways that L can be attached to $\partial(T^2$ cut C). In the first case, the end points of L are copies of each other. This is shown in diagram 6.A.1. In order for $T^2-f^{-1}(y)$ to have exactly two components, $f^{-1}(y)$ must be $\partial((T^2$ cut C) cut L) together with an arc or a circle M in $((T^2$ cut C) cut L) attached to $\partial((T^2$ cut C) cut L). To ensure that in T^2, $f^{-1}(y)$ has an even number of branches meeting at each point, M in $((T^2$ cut C) cut L) must be attached at each end to the points a, b, c and d or at neither end. In this context, a circle may be considered an arc with both ends joined. Diagram 6.A.3 lists the distinct ways this attachment can be done.

The other possibility for L is that its end points are not copies of each other as shown in diagram 6.B.2, where e is the copy of a in $(T^2$ cut C), and k the copy of d in $(T^2$ cut C). Once again, the remaining portion of $f^{-1}(y)$ aside from C and L must be an arc or a circle in $(T^2$ cut C). However, because $f^{-1}(y)$ has an even number of branches at each point, this portion must be an arc, M, joined at one end to either a, c or e, and at the other end to b, d or k. Diagram 6.B.3 lists the distinct ways this can be done.

The proof of topological equivalence in all these cases parallels the others. Cases in which M is an arc are handled as in

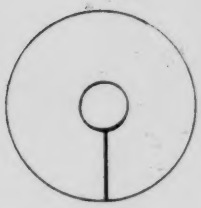
case (i) for the projective plane. Cases in which M is a circle are handled in the fashion of case (ii) for the projective plane.

Finally, it can be seen from the level curves that cases VI and VI', cases VIII and VIII', and the cases IX and IX' are topologically equivalent.

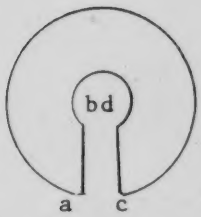
Examples of these functions are listed in the Appendix in the order presented here. □

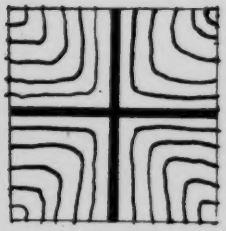
DIAGRAM 6.A

1



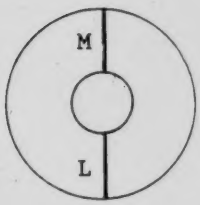
2





3

III



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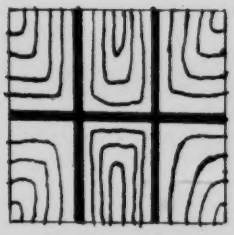

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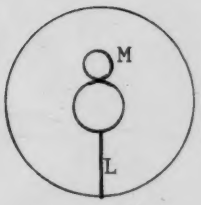

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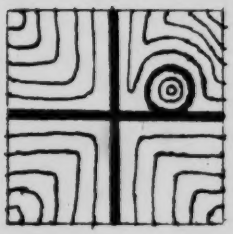

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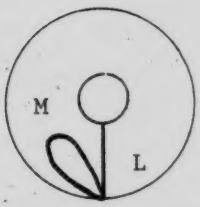


a,a

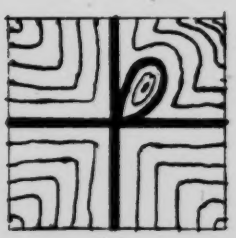
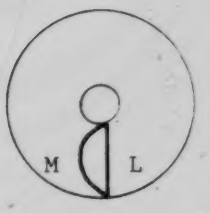
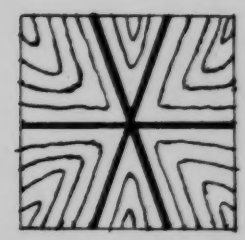


Diagram 6.A - continued

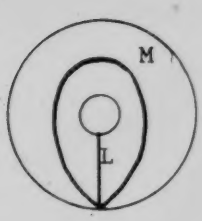
VI



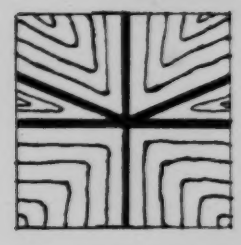
a,b



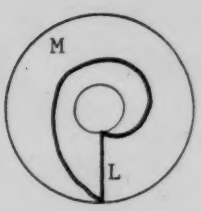
VI'



a,c



VII



a,d

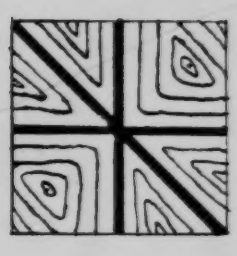
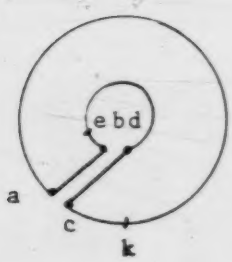


DIAGRAM 6.B

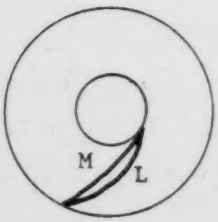
1

2

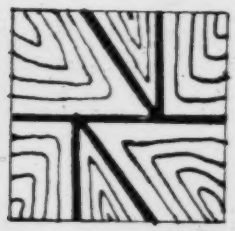


3

VIII



a,b



IX

a,d

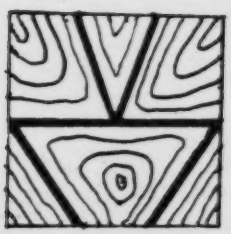


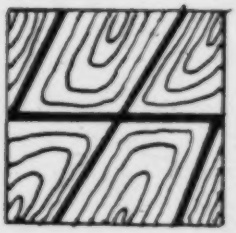
Diagram 6.B - continued

VIII'

a,k

IX'

e,k



CHAPTER VII

POSSIBLE EXTENSIONS

The classification of real valued monotone analytic functions of S^2, P^2 and T^2 raises the question of whether similar techniques will enable these functions to be classified on arbitrary surfaces without boundary. Because every such surface can be characterized as either a sphere, a connected sum of projective planes or a connected sum of tori, the general classification on surfaces becomes a natural problem.

Another problem that arises is the classification of monotone analytic functions on surfaces where the functions take values in the circle, S^1. Of course, this problem will only be distinct from the real valued classification problem when the functions studied map onto S^1. One result can be stated immediately.

Prop. 7.1 There exist no monotone C^ω functions on either S^2 or P^2 which map onto S^1. Any such function on T^2 is trivial.

Pf. Suppose f is a monotone C^ω function from S^2 onto S^1. Then there exists some $x \in S^1$ such that $f^{-1}(x)$ is a circle. However, $f^{-1}(x)$ cannot disconnect S^2 by Prop. 2.1. This contradicts Prop. 5.2.

If f is a monotone C^ω function on P^2 onto S^1 then there exists an $x \in S^1$ such that $f^{-1}(x)$ is a circle. Then P^2 cut $f^{-1}(x)$) is a disc, contradicting Prop. 5.2.

If f is a monotone C^ω function on T^2 onto S^1, then for every $x \in S^1$, $f^{-1}(x)$ is either a point or a circle. For if $f^{-1}(x)$ were the connected union of two circles C_1, C_2 then

$((T^2$ cut $C_1)$ cut $C_2)$ would be a disc, contradicting Prop. 5.2.

However, the inverse image of every point must be a circle. If this were not the case, and $f^{-1}(y)$ were a point, then by Lemma 4.3 (a), there would be some x close to y such that $f^{-1}(x)$ would be a circle disconnecting T^2 (because it could have arbitrarily small diameter). This is a contradiction. The complete regularity of f follows from Prop. 6.2 applied to $f^{-1}(I)$, where I is any closed arc of S^1. Kim [10] shows that f must be trivial. □

The techniques developed in this thesis are not sufficient for higher dimensional classification theorems. The primary difficulty in higher dimensions is that compact connected manifolds have not yet been classified in dimensions greater than two. The Schoenflies Theorem cannot be generalized in its full strength to dimension three. It must be modified to include local flatness before it will hold in higher dimensions. See, for example, Rushing [17]. The notion of 0-regularity can be generalized to the notion of n-regularity. Theorems relating n-regularity and complete regularity exist for $n \leq 2$ (Hamstrom [8]).

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APPENDIX

EXAMPLES OF MONOTONE MAPPINGS ON S^2, P^2, T^2

S1, P1, T1. Let X be a compact connected topological space, and c a real number. Define $f: X \to \mathbb{R}$ by $f(X) = c$.

S2. Define $f: S^2 \to \mathbb{R}$ by $f(x,y,z) = x$.

P2. Let $f: P^2 \to \mathbb{R}$ be the function induced by the quotient map $q: S^2 \to P^2$ and the function $g: S^2 \to \mathbb{R}$ defined by $g(x,y,z) = z^2$.

P3. Let $f: P^2 \to \mathbb{R}$ be induced as above from $g(x,y,z) = xz$.

P4. Let $f: P^2 \to \mathbb{R}$ be induced by $g(x,y,z) = (x^2 + y^2 - z^2)(z - x)^2$.

T2. Parameterize T^2 as (θ,ϕ), where (θ,ϕ) corresponds to $(\cos\theta, \sin\theta, \cos\phi, \sin\phi) \in \mathbb{R}^4$. Let $f: T^2 \to \mathbb{R}$ be defined by $f(\theta,\phi) = (1 + \sin\theta)(1 + \sin\phi)$.

T3. Define $f: T^2 \to \mathbb{R}$ by $f(\theta,\phi) = \cos\theta(1 + \sin\phi)$.

T4. Define $f: T^2 \to \mathbb{R}$ by

$$f(\theta,\phi) = \sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\phi}{2}\right)\left[\sin^2\left(\frac{\theta}{2} - \frac{\pi}{4}\right) + \sin^2\left(\frac{\phi-\epsilon}{2}\right) - \sin^2\left(\frac{\epsilon}{2}\right)\right]$$

for $\epsilon > 0$ sufficiently small.

T5. Define $f: T^2 \to \mathbb{R}$ by

$$f(\theta,\phi) = \sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\phi}{2}\right)\left\{\sin^2\left(\frac{\theta-\phi}{4}\right) - \left[\sin^2\epsilon - \sin^2\left(\frac{\theta+\phi}{4} - \epsilon\right)\right]^3\right\}$$

for $\epsilon > 0$ sufficiently small.

T6. Define $f: T^2 \to \mathbb{R}$ by

$$f(\theta,\phi) = [\sin^2(\tfrac{\theta}{2}) - \sin^2(\tfrac{\phi}{2}) \cos\theta]\sin^2\tfrac{\phi}{2} \ .$$

T7. No function found.

T8. Define $f: T^2 \to \mathbb{R}$ by

$$f(\theta,\phi) = [1 - \cos(m\phi - \theta - \tfrac{\pi}{m})][\sin^2(\tfrac{\theta}{2}) + \sin^2(\tfrac{\phi}{2}) - \varepsilon]$$

where m is a positive integer greater than 1 and $\varepsilon > 0$ is chosen to be the smallest value so that $\cos(m\phi - \theta - \frac{\pi}{m}) = 1$ and $\sin^2(\frac{\theta}{2}) + \sin^2(\frac{\phi}{2}) = \varepsilon$ have precisely two distinct simultaneous solutions.

T9. Define $f: T^2 \to \mathbb{R}$ by $f(\theta,\phi) = \sin\theta + \sin\phi$.